Concavity, Core-concavity, Quasiconcavity: A Generalizing Framework for Entropy Measures.

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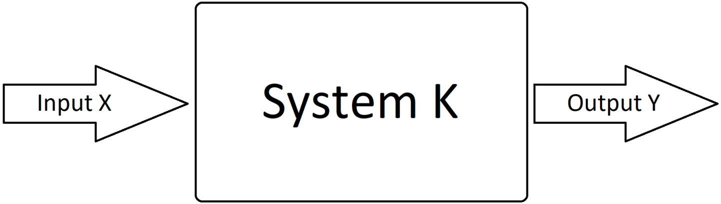
34th IEEE Computer Security Foundations Symposium

23 June 2021

Introduction

Introduction: The Basic Setting



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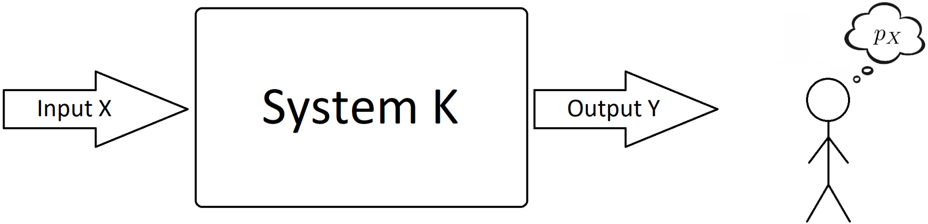

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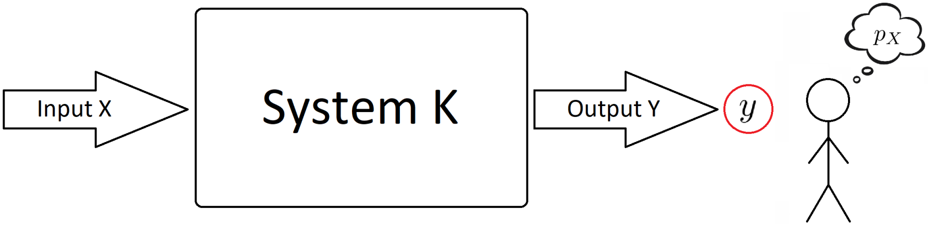

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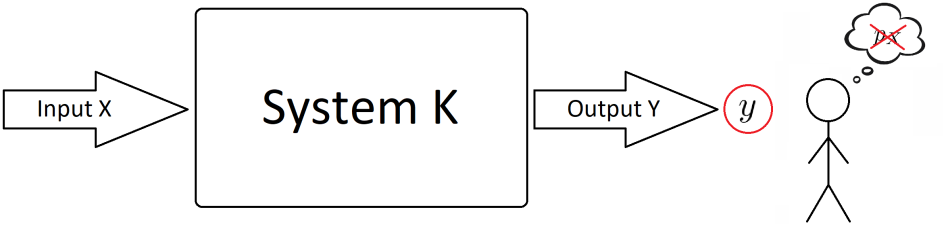

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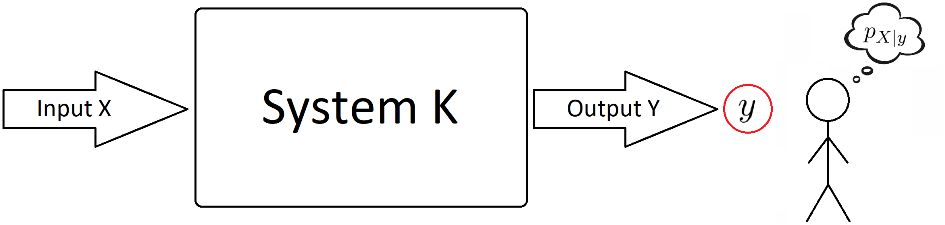

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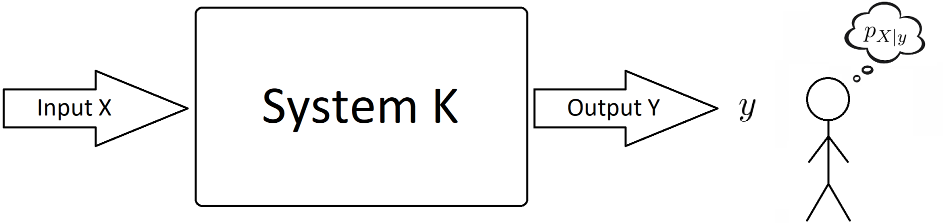

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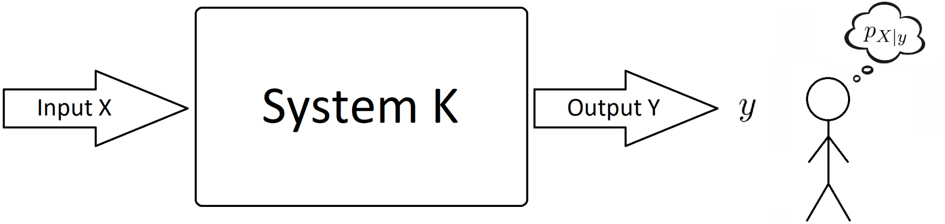

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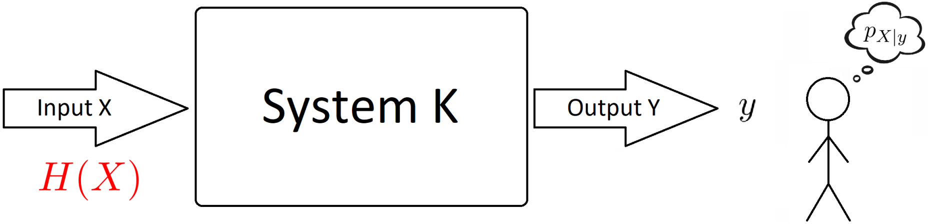

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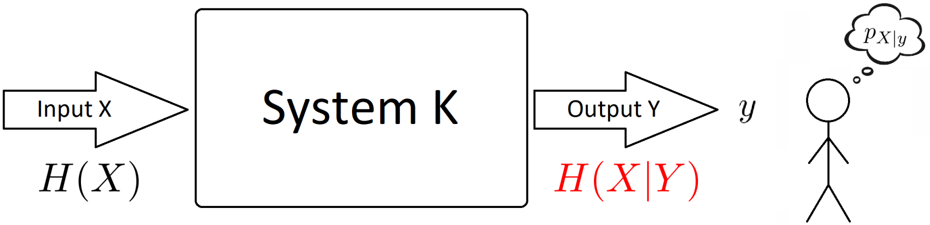

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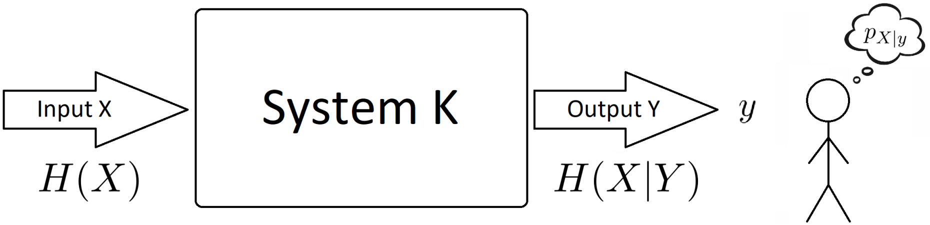

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- Information Leakage: $I_H(X;Y) = H(X) - H(X|Y)$

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- Minimum (`MIN`):

$$H(X|Y) = \min_y H(X|y)$$

  - A worst-case scenario: useful when large leakage is unacceptable, even if unlikely (e.g. privacy)

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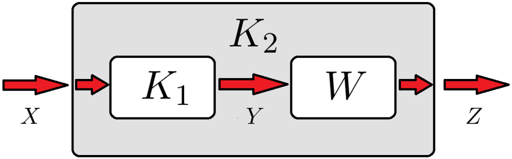

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- We denote the family of entropies satisfying $\texttt{EAVG}$ and $\texttt{CCV}$ by $\mathcal{H}_{\texttt{EAVG}}$.
- Notice that $\mathcal{C}_{\texttt{AVG}} \subset \mathcal{H}_{\texttt{EAVG}}$, by taking $\eta = \mathrm{id}$.

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- Yes, we can!

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- This has been first discovered by Bruno de Finetti in the paper *Sulle stratificazioni convesse* (1949), motivated by the study of utility functions in microeconomics

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- By combining these results, we were able to prove that $\mathcal{Q}_{\texttt{MIN}} \subset \mathcal{Q}$

Applications

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  - A bound in terms of probability of error, that generalises Fano's inequality:

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where $\hat{e} = \sum_y p(y)(1 - \max_x p(x|y))$

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  - Are there other families with interesting conditional forms to be derived from $\mathcal{Q}$?

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